TECHNICAL NOTE 32

The quartz crystal model and its frequencies

1. Introduction

In this note, we present some of the basic electrical properties of quartz crystals. In particular, we present the 4-parameter crystal model, examine its resonant and antiresonant frequencies, and determine the frequency at load capacitance. Our coverage is brief, yet complete enough to cover most cases of practical interest. For further information, the interested reader should consult References [1] and [2]. The model and analysis is applicable to most types of quartz crystals, in particular tuning-fork, extensional-mode, and AT-cut resonators.

1.1 Overview

To begin, let's look at the impedance of a real 20 MHz crystal around its fundamental mode.

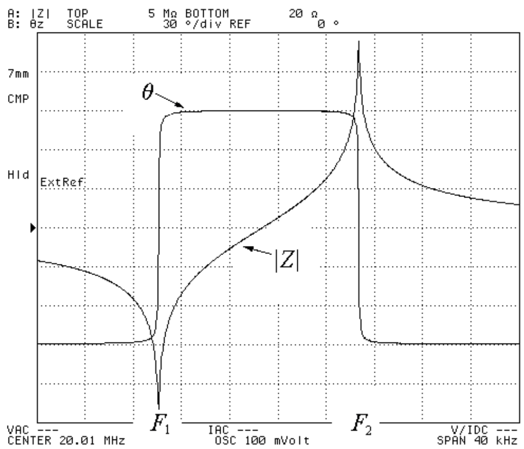

Figure 1—Impedance magnitude |Z| (log scale) and phase θ versus frequency for an approximately 20 MHz crystal. (Scans made with an Agilent 4294A Impedance Analyzer.)

In this impedance scan over frequency (Figure 1), we see the following qualitative behavior. There are two frequencies F_1 and F_2 where the phase θ is zero. Below and away from F_1, the phase is approximately -90°. Near F_1 the phase makes a fast transition from -90° to +90°. Between F_1 and F_2 the phase remains approximately constant at +90°. Near F_2 the phase makes a fast transition from +90° to -90°. Lastly, above and away from F_2, the phase is again approximately -90°. Further, the impedance of the crystal is least at F_1 and greatest at F_2.

The region between F_1 and F_2 is a region of positive reactance, and hence is called the inductive region. For a given AC voltage across the crystal, the net current flow through the crystal is greatest at F_1 and least at F_2. In loose terms, F_1 is referred to as the series-resonant frequency and F_2 is referred to as the parallel-resonant frequency (also called antiresonance).

Likewise, we can express the impedance in terms of its resistance (real part) and reactance (imaginary part) as shown in Figure 2.

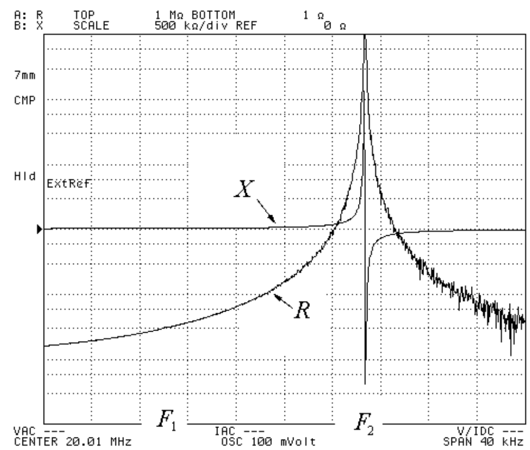

Figure 2—Impedance resistance *R* (log scale) and reactance *X* versus frequency for the same crystal shown in Figure 1.

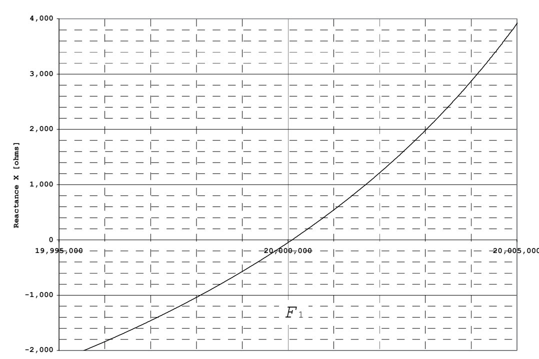

Figure 3—Close-up of reactance *X* near F_1. The reactance is zero at a frequency slightly above 20 MHz.

The resistance R is strongly peaked at the frequency F_2. Below F_1, the reactance is negative and increases to zero at F_1 (see Figure 3) and then increases to large positive values as F_2 is approached. At F_2, the reactance quickly decreases to large negative values and then again steadily increases towards zero.

1.2 The crystal frequencies

1.2.1 The series-resonant frequencies

Consider the frequency F_1. One can define this in at least three ways. One choice is the (lower) frequency F_r where the phase of the crystal is zero. At this zero-phase frequency, the crystal is purely resistive (equivalently its reactance is zero). A second choice is the frequency F_m of minimum impedance. A third choice is to define this as the series-resonant frequency F_s—a frequency whose definition requires the crystal model as discussed in Section 2.

Table 1—The series-resonant frequencies

Frequency	*Description*
F_s	Series resonant frequency
F_r	Zero-phase frequency (lower)
F_m	Minimum impedance frequency

It turns out that for most crystals, F_s, F_r, and F_m are all sufficiently close to one another than it is not necessary to distinguish between them.

$$F_s \approx F_r \approx F_m \,. \tag{1}$$

See Section 6.2 for further details.

1.2.2 The parallel-resonant frequencies

Next consider the frequency F_2. One can also define this in at least three ways. One choice is the (upper) frequency F_a where the phase of the crystal is zero. At this zero-phase frequency, the crystal is purely resistive (being very high). A second choice is the frequency F_n of maximum impedance. A third choice is to define this as the parallel-resonant frequency F_p—a frequency whose definition also requires the crystal model as discussed in Section 2.

Table 2—The parallel-resonant frequencies

Frequency	*Description*
F_p	Parallel-resonant frequency
F_a	Zero-phase frequency (upper)
F_n	Maximum-impedance frequency

For most crystals, F_p, F_a, and F_n are all sufficiently close to one another than it is not necessary to distinguish between them.

$$F_p \approx F_a \approx F_n \,. \tag{2}$$

Further, they are above F_s and are normally well approximated by the expression

$$F_p \approx F_s\left(1+\frac{C_1}{2C_0}\right). \tag{3}$$

1.2.3 Frequency at load capacitance

Another important crystal frequency is the frequency F_L at a load capacitance C_L. (See Reference [3] for a full discussion of this concept.) At this frequency, the crystal reactance X is equal to

$$X = \frac{1}{\omega C_L}, \tag{4}$$

where $\omega = 2\pi F_L$. Equivalently, at this frequency, the series combination of the crystal and a capacitance C_L has zero reactance. (See Figure 4.) Note that as $C_L \to \infty$, $F_L \to F_r$, and that as C_L decreases, F_L increases towards F_p.

Table 3—Frequency at load capacitance

Frequency	*Description*
F_L	Frequency at load capacitance

The resulting relation giving the frequency of a crystal as a function of its parameters and a load capacitance C_L is called the *crystal-frequency equation* and is of prime importance in specifying and understanding the operation of crystals in oscillators. (See References [3] and [4].)

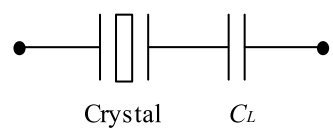

Figure 4—Defining F_L at C_L

As we shall see, for most applications, the frequency F_L at load capacitance C_L is well approximated by the expression

$$F_L \approx F_s\left(1+\frac{C_1}{2(C_0+C_L)}\right). \tag{5}$$

1.3 Guide to this note

In Section 2, we present and discuss the 4-parameter crystal model. In Section 3, we derive some simple results from this model defining F_s and F_p. In Section 4, we define the three non-dimensional quantities r, Q, and M. In Section 5, we present some useful properties of the frequencies F_r, F_a, F_s, and F_p. In Section 6, we present approximations for F_L and F_r that go beyond the approximations in Section 1.2. In Section 7, we derive the exact expressions for F_L and F_r. In Section 8, we make a few comments on resistance at resonance and antiresonance. Lastly, Appendix 1 contains a list of the important symbols used in this note.

Note that while we present both exact and approximate relations for F_s, F_r, F_p, F_a, and F_L, we present no further results for F_m or F_n other than the approximations given in Section 1.2. For further information, see References [5], [6], and [7].

For those first becoming acquainted with crystals, we recommend reading Sections 1-4. For those who want further details and more precise results, we recommend reading Sections 1-6. Lastly, for those who want exact results, we recommend this entire note.

Throughout, we use the usual relation between a given frequency f and its angular frequency ω counterpart

$$\omega = 2\pi f \ . \tag{6}$$

2. The 4-parameter crystal model

The modes of interest in quartz crystals are usually modeled electrically by the 4-parameter model shown in Figure 5. This model consists of two arms in parallel with one another. The "static arm" consists of a single capacitance C_0 (also referred to as the shunt capacitance). Herein, this capacitance includes the capacitance of the bare crystal and the shunt capacitance of its packaging. The "motional arm" consists of the series combination of a resistance R_1, inductance L_1, and a capacitance C_1.

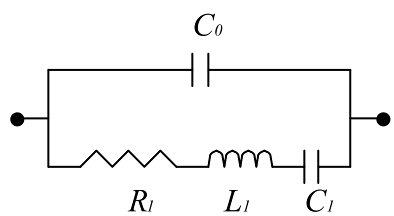

Figure 5—The 4-parameter crystal model

While this model is an approximation of the electrical characteristics of the crystal, it is a very good one and for most purposes more than sufficient. So, from here on, we take this model seriously. (See References [1] and [8] for further discussion.)

One should be aware that crystals are complicated by the existence of many modes of oscillation. In addition to their fundamental mode, crystals have overtone modes. For example, tuning-fork crystals have 1st-overtone modes at roughly six times the frequency of the fundamental mode. As another example, AT-cut crystals have 3rd, 5th, 7th, etc., overtone modes with frequencies being nearly the overtone number times the fundamental mode frequency. Sometimes these modes are the desired mode as they can offer frequencies that would otherwise be unattainable.[1] When they are not the desired mode, their great separation from the main mode and their resistance is normally sufficiently high that they have no effect on the performance of the crystal in an oscillator. AT-cut crystals are further complicated by the existence of anharmonic modes just above the main mode as well as having other unwanted modes. Proper crystal design minimizes the strengths of these modes (collectively referred to as unwanted modes) so that they have no effect on the crystal's operation in an oscillator.

2.1 *Typical values*

To give the reader some idea of these crystal parameters and how they vary with crystal type and frequency, we present some typical values for Statek crystals. However, keep in mind that ranges given here can be exceeded in some cases.

The static capacitance C_0 has a limited range of variation, usually being on the order of 1-3 pF. This parameter typically scales with the motional capacitance C_1 and package size, i.e. crystals with large C_1 in large packages have large C_0. Smaller crystals also tend to have smaller C_1, so C_0 roughly correlates with package size, but not absolutely.

Similarly, the motional capacitance C_1 has a fairly limited range typically being on the order of 0.5 fF to 10 fF. Tuning-fork and extensional-mode crystals tend to have their C_1 lie on the low end of this spectrum, while AT-cut crystals can have a C_1 just about anywhere in this range, depending on the size of the crystal and its frequency.

The motional inductance L_1 varies greatly over frequency, for as we shall see, it is determined by C_1 and the crystal frequency. It has a high of roughly 100 kH for 10 kHz crystals to a low of less than 1 mH for 100 MHz crystals (a range of about 10^8).

Lastly, the crystal resistance also varies greatly over frequency from a high of about 1 MΩ for 10 kHz H-type crystals to a low of about 10 Ω for high-frequency AT-cut crystals (a range of about 10^5).

2.2 *Specifying crystal parameters*

If your application has critical requirements that require specification of the crystal parameters, then bounds on the relevant parameters should be supplied. However, unnecessary requirements will probably increase the cost of the crystal without any added benefit.

The crystal parameter that most commonly requires specification is the crystal resistance R_1. This parameter plays an important role in the crystal-oscillator gain requirement and sometimes an upper

[1] It is normally a mistake to use a crystal designed for fundamental mode operation at one of its overtone modes.

bound on R_1 is required to ensure the startup of the oscillator.

For applications requiring crystal pullability (the ability to change frequency with changes in load capacitance), bounds should be placed on C_1. As shown by Equation (5), C_1 plays a primary role in determining the frequency change for a given change in C_L. The capacitance C_0 also plays a role and when the pullability requirements are demanding, upper bounds are placed on C_0.

Lastly, requirements on L_1 are rarely necessary as such conditions can be expressed as conditions on C_1. [See Equation (14).]

Sometimes people place requirements on the crystal Q (defined below) in the belief that this quantity determines either oscillator startup or crystal pullability. Both are wrong. The resistance R_1 (along with the oscillator design) determines startup. The motional capacitance C_1 (along with C_0) determines the crystal pullability. [See Equation (5).]

3. Simple consequences

In this section we derive some simple consequences of the crystal model. In particular, we show the existence of a series-resonant frequency F_s and a parallel-resonant frequency F_p.

The impedance Z of the crystal is determined by the parallel combination of the impedance Z_0 of the static arm and the impedance Z_1 of the motional arm.

$$Z = \frac{Z_0 Z_1}{Z_0 + Z_1} . \tag{7}$$

The impedance of the static arm is purely reactive and is given by

$$Z_0 = jX_0 , \tag{8}$$

where its reactance X_0 is given by

$$X_0 = -\frac{1}{\omega C_0} . \tag{9}$$

Likewise, the impedance of the motional arm is given by

$$Z_1 = R_1 + jX_1 , \tag{10}$$

where its reactance X_1 is given by

$$X_1 = \omega L_1 - \frac{1}{\omega C_1} . \tag{11}$$

3.1 Series resonance

Being a capacitance, the reactance of the static arm is negative. On the other hand, the motional arm consisting of the series combination an inductor and a capacitor can have reactance of either sign depending on the frequency. In particular at some frequency F_s, called the series-resonant frequency,

$$X_1 = 0 . \tag{12}$$

Using Equation (11), we see that the angular frequency ω_s at which the reactance of the motional arm is zero is given by

$$\omega_s = \frac{1}{\sqrt{L_1 C_1}} . \tag{13}$$

Therefore, the series-resonant frequency of the crystal is given by

$$F_s = \frac{1}{2\pi} \frac{1}{\sqrt{L_1 C_1}} . \tag{14}$$

Equivalently, ignoring the crystal resistance R_1, series resonance is the frequency at which the crystal impedance is minimal (being zero in this idealization).

Note that one should not use Equation (14) to compute F_s as typically neither L_1 and C_1 are known only to about 1% accuracy while other methods can determine F_s to better than 1 ppm. Instead, the utility of Equation (14) comes in computing either L_1 or C_1 from the other and F_s.

3.2 Parallel resonance

One effect of the static (shunt) capacitance C_0 is to make the crystal look like a simple capacitance at frequencies where the impedance of the motional arm is large compared to impedance of the static arm. Another is to create an anti-resonance (resonance of high impedance) at a frequency where the two arms of the crystal resonant in which such a way to offer high impedance to current flow.

Ignoring the crystal resistance R_1, this parallel resonance occurs at the frequency where the admittance $Y = 1/Z$ of the crystal is zero.

$$\begin{aligned} 0 &= Y \\ &= Y_0 + Y_1 \\ &= \frac{1}{jX_0} + \frac{1}{jX_1} . \end{aligned} \tag{15}$$

Therefore

$$X_0 + X_1 = 0 , \tag{16}$$

or equivalently

$$X_1 = \frac{1}{\omega C_0}\,. \tag{17}$$

With this, it follows that the parallel-resonant frequency F_p of the crystal is given by

$$F_p = F_s\sqrt{1+\frac{C_1}{C_0}}\,. \tag{18}$$

Note that the parallel resonant frequency is always above the series-resonant frequency and that their separation is determined by the ratio of the capacitances C_1 and C_0. For quartz crystals, $C_1 << C_0$, so F_s and F_p are quite close is as fraction of absolute frequency and is usually well approximated by the expression

$$F_p \approx F_s\left(1+\frac{C_1}{2C_0}\right). \tag{19}$$

While the motivation for our definition of the parallel-resonant frequency was based on the case where the crystal resistance is zero, we take its definition in general to be that frequency where the reactances of the two arms are in anti-resonance. Therefore, the parallel resonant frequency F_p of a crystal is always given by Equation (18).

3.3 F_L at C_L

Ignoring the crystal resistance R_1, we can easily work out the crystal frequency F_L at a load capacitance C_L. This is frequency at which

$$\begin{aligned} \frac{1}{Z} &= Y_0 + Y_1 \\ \frac{1}{(j/\omega C_L)} &= \frac{1}{(-j/\omega C_0)} + \frac{1}{jX_1}, \end{aligned} \tag{20}$$

and so

$$X_1 = \frac{1}{\omega(C_0 + C_L)}. \tag{21}$$

With this is it straightforward to show that the frequency F_L at load capacitance C_L is given by

$$F_L = F_s\sqrt{1+\frac{C_1}{C_0+C_L}},\quad (R_1 = 0)\,. \tag{22}$$

Although this derivation ignores the crystal resistance, our final expression is sufficient for most applications and in fact is normally further approximated by the expression

$$F_L \approx F_s\left(1+\frac{C_1}{2(C_0+C_L)}\right). \tag{23}$$

This is the standard crystal-frequency equation. However, be aware that it is an approximation. Even so, in most cases this equation is sufficient and a more exact expression would complicate the computation without any benefit.

3.4 The significance of L_1

Note that

$$\begin{aligned} \frac{dX_1}{df} &= 2\pi\frac{dX_1}{d\omega} \\ &= 2\pi\left(L_1 + \frac{1}{\omega^2 C_1}\right) \\ &= 4\pi L_1 \quad \text{at } f = F_s, \end{aligned} \tag{24}$$

which shows that L_1 is proportional to the rate of change of the motional reactance with frequency at series resonance. This fact is sometimes useful in measuring the crystal parameters. Note that, ignoring the effects of resistance, $dX/df = 4\pi L_1$ at $f = F_s$, showing that the shunt capacitance C_0 does not modify the slope of reactance curve at series resonance. However, the shunt capacitance does greatly increase the slope of the reactance as antiresonance is approached. (See Figure 2 and Figure 3.)

4. Three non-dimensional quantities

There are at least three non-dimensional quantities that are very useful in characterizing crystals.

Our first quantity is the capacitance ratio r

$$r = \frac{C_0}{C_1}\,. \tag{25}$$

As we saw in Section 3.2, $1/r$ determines the separation between series resonance and parallel resonance, in other words, the width of the crystal's inductive region. As an example, a crystal with a C_0 of 2 pF and a C_1 of 5 fF has a capacitance ratio of 400. For Statek crystals, r can range from about 250 to 1,000.[2]

Our next quantity is the crystal quality factor Q. This is defined so that $2\pi/Q$ is the fractional energy lost per cycle in the crystal and is given in terms of the crystal parameters by

$$Q = \frac{1}{\omega_s R_1 C_1}. \quad (26)$$

where the frequency ω_s (angular series resonance) is given by Equation (13). Crystals with large Q oscillate many cycles before their oscillations decay appreciably. For Statek crystals, Q ranges from about 2,000 to 400,000.[2]

A direct consequence of its definition is that it takes

$$\frac{Q}{2\pi} \quad (27)$$

cycles for the oscillation energy an isolated crystal to ramp-down by a factor of $1/e$. The number of cycles for ramp-up is the same. So, the time for oscillations to ramp-up or ramp-down in a low-frequency high-Q crystal can be quite long—on the order of seconds.

Our last quantity is the crystal figure-of-merit M. This is simply the ratio of the impedance of the static arm to the impedance of the motional arm at series resonance. Given this, it is straightforward to show that M is given by

$$M = \frac{1}{\omega_s R_1 C_0}. \quad (28)$$

As we shall show in Section 7.3, in order for the crystal to posses an inductive region, M be greater than 2. For Statek crystals, M ranges from about 10 to 300.[2]

Note our three parameters are not independent; indeed

$$Q = Mr. \quad (29)$$

5. Some useful frequency properties

5.1 The frequency product property

It turns out, without approximation, that

$$F_r F_a = F_s F_p. \quad (30)$$

This equality allows us to calculate any one of the above four frequencies given the other three. Because of this and the fact that results beyond the approximation $F_a \approx F_p$ are rarely required, we do not present any further expressions for F_a.

5.2 The frequency inequalities

When the crystal posses an inductive region (so F_r and F_a exist and are distinct)

$$F_s \le F_r < F_a \le F_p, \quad (31)$$

with equalities when and only when $R_1 = 0$.

6. Approximations beyond $R_1 = 0$

For the zero-phase frequency F_r, the frequency F_L at load capacitance C_L, and crystal resistance at load capacitance, we simply present approximate expressions that go beyond the results presented so far. For proof, see the exact results in Section 7.

6.1 Approximating F_L at C_L

In cases where further accuracy is required in the load frequency F_L, then to second order in resistance

$$F_L \approx F_s \sqrt{\left(1 + \frac{C_1}{C_0 + C_L}\right)\left(1 + \frac{1}{QM}\left(1 + \frac{C_0}{C_L}\right)\right)}. \quad (32)$$

6.2 Approximating F_r

Normally, F_r is well approximated by F_s. To see the difference between the two, we must look to second order in crystal resistance. To this order, F_r is slightly above F_s by the amount

$$F_r \approx F_s\left(1 + \frac{1}{2QM}\right). \quad (33)$$

This result can be obtained from Equation (32) by taking the limit $C_L \to \infty$ and performing a first-order expansion in $1/(QM)$.

In most cases, $QM > 10^6$ so that ignoring the effect of resistance is to make an error below 1 part-per-million in frequency, which is usually acceptable. An interesting exception is the case of 10 kHz H-type crystals. The large resistance (about 1 MΩ) of these crystals give $Q \approx 4{,}000$ and a figure-of-merit $M \approx 11$ and so that the difference between F_r and F_s is about 11 ppm.

6.3 Approximating R at F_L

The crystal resistance R depends on the frequency of interest. At series resonance, $R \approx R_1$ and it increases to very large values near parallel resonance. (See Figure 2.) A natural question that comes up is what is the crystal resistance at the load frequency F_L. It turns out to good approximation that

$$R \approx R_1\left(1 + \frac{C_0}{C_L}\right)^2. \quad (34)$$

So, as expected, the crystal resistance is approximately R_1 at F_r and increases to very large values as C_L approaches zero.

7. Exact expressions

7.1 Crystal impedance

We express the crystal impedance Z in terms of the impedances of the two parallel arms as follows

$$\begin{aligned} Z &= \frac{Z_0 Z_1}{Z_0 + Z_1} \\ &= \frac{Z_1 |Z_0|^2 + Z_0 |Z_1|^2}{|Z_0 + Z_1|^2}. \end{aligned} \quad (35)$$

Denoting resistance of the crystal by R ($R = \mathrm{Re}(Z)$) and its reactance by X ($X = \mathrm{Im}(Z)$), so that

$$Z = R + jX ,$$

then Equations (8) and (10), give the following expression for the crystal resistance R

$$R = R_1 \left(\frac{X_0^2}{R_1^2 + (X_0 + X_1)^2} \right), \quad (36)$$

and the following for the crystal reactance X

$$\begin{aligned} X &= X_0 \left(\frac{R_1^2 + X_1 X_0 + X_1^2}{R_1^2 + (X_0 + X_1)^2} \right) \\ &= X_0 \left(1 - X_0 \left(\frac{X_0 + X_1}{R_1^2 + (X_0 + X_1)^2} \right) \right). \end{aligned} \quad (37)$$

It can be shown that $Z(\omega)$ sweeps out an approximate circle in the impedance plane (and similarly $Y(\omega)$ sweeps out an approximate circle in the admittance plane). See References [5], [6], and [7] for further details.

7.2 Normalization

Define the normalized "frequency" Ω by

$$\Omega = \frac{\omega^2 - \omega_s^2}{\omega_p^2 - \omega_s^2}. \quad (38)$$

Note that $\Omega = 0$ at F_s and $\Omega = 1$ at F_p. In terms of Ω, the (angular) frequency is

$$\omega = \omega_s \sqrt{1 + \Omega / r}. \quad (39)$$

Further set

$$\overline{\Omega} = 1 - \Omega. \quad (40)$$

Note that

$$\Omega = -\frac{X_1}{X_0}, \quad (41)$$

and

$$\overline{\Omega} = \frac{X_0 + X_1}{X_0}. \quad (42)$$

With these definitions, we can express the crystal resistance R as

$$R = R_1 \left(\frac{1}{\left(1 + \left(1 - \overline{\Omega}\right)/r\right) M^{-2} + \overline{\Omega}^2} \right), \quad (43)$$

and the reactance as

$$\frac{X}{X_0} = 1 - \frac{\overline{\Omega}}{\left(1 + \left(1 - \overline{\Omega}\right)/r\right) M^{-2} + \overline{\Omega}^2}. \quad (44)$$

7.3 F_L at C_L (exact)

Using Equation (44) and the fact that $X = 1/(\omega C_L)$, we have

$$-\frac{C_0}{C_L} = 1 - \frac{\overline{\Omega}}{\left(1 + \left(1 - \overline{\Omega}\right)/r\right) M^{-2} + \overline{\Omega}^2}, \quad (45)$$

which gives us the following quadratic equation for $\overline{\Omega}$

$$\overline{\Omega}^2 - \left(\frac{1}{1 + C_0/C_L} + \frac{1}{QM} \right) \overline{\Omega} + \frac{1}{M^2} \left(1 + \frac{1}{r} \right) = 0. \quad (46)$$

It can be shown that the existence of real frequencies requires that

$$M > \left(2\sqrt{1 + \frac{C_1}{C_0 + C_L}} + \frac{1}{Q} \right) \left(1 + \frac{C_0}{C_L} \right). \quad (47)$$

Alternatively, for a given crystal, there is a lower bound on the allowed load capacitance C_L, i.e. arbitrarily small load capacitances are not allowed. However, this bound is usually so weak that it causes no practical limitation to the existence of F_L.

Taking the limit $C_L \to \infty$, we see that the condition for the existence of an inductive region is that

$$M > 2 + \frac{1}{Q}, \quad (48)$$

showing that M must be greater than 2.

Define the quantity ξ by

$$\xi = \frac{1}{1 + C_0/C_L} - \frac{1}{QM}. \quad (49)$$

Note that $\xi > 0$ by Equation (47). Further, define the quantity χ by

$$\chi = \frac{2}{\xi + \sqrt{\xi^2 - \frac{4}{M^2}\left(1 + \frac{C_1}{C_0 + C_L}\right)}} . \qquad (50)$$

With these definitions, solving Equation (46) and using Equation (39), the frequency F_L at load capacitance C_L is given by

$$F_L = F_s \sqrt{\left(1 + \frac{C_1}{C_0 + C_L}\right)\left(1 + \frac{\chi}{QM}\right)} . \qquad (51)$$

Note that actually there is a second root of Equation (46). However, the frequency corresponding to this root corresponds to a frequency very near anti-resonance where the resistance is very high; this is not the frequency of interest.

7.4 F_r (exact)

Taking the limit $C_L \to \infty$ in the above equation for F_L gives F_r (the lower frequency at which the reactance of the crystal is zero)

$$F_r = F_s \sqrt{1 + \frac{1}{QM}\left(\frac{2}{\left(1 - \frac{1}{QM}\right) + \sqrt{\left(1 - \frac{1}{QM}\right)^2 - \frac{4}{M^2}}}\right)} \qquad (52)$$

8. Final comments on crystal resistance

Note that the crystal's resistance at neither F_s nor F_r is R_1. Instead,

$$R(F_s) = \frac{R_1}{1 + M^{-2}} < R_1 , \qquad (53)$$

and

$$R(F_r) \approx \frac{R_1}{1 - M^{-2}} > R_1 . \qquad (54)$$

However, normally such distinctions are not required as they differ from R_1 by much less than 1%.

Lastly, near antiresonance,

$$R(F_p) \approx R_1 M^2 , \qquad (55)$$

showing that M^2 is roughly the range of the crystal resistance over frequency. (In fact, M^2 is also roughly the range of the crystal impedance over frequency.)

9. References

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10. Appendix 1—Table of symbols

Table 4—Symbols used to describe crystals

Symbol	*Alternates*	*Description*
f		Frequency
ω		Angular frequency, $\omega = 2\pi f$
C_0		Static (shunt) capacitance
R_1	R_m, *R1*, *RR*[3]	Motional resistance
L_1	L_m, *L1*	Motional inductance
C_1	C_m, *C1*	Motional capacitance
r		Capacitance ratio, $r = C_0/C_1$
Q		Resonator quality factor, $Q = 1/(\omega_s R_1 C_1)$
M		Figure of merit, $M = 1/(\omega_s R_1 C_1) = Q/r$
F_s		Series resonant frequency
F_r	*FR*	Lower zero-phase frequency (normally close to F_s)
F_m		Minimum impedance frequency (normally close to F_s)
F_p		Parallel resonant frequency
F_a		Upper zero-phase frequency (normally close to F_p)
F_n		Maximum impedance frequency (normally close to F_p)
F_L		Frequency at load capacitance C_L
C_L	*CL*	Load capacitance
TS		Trim sensitivity, fractional rate-of-change of F_L with C_L
Z		Crystal impedance, $Z = R + jX = \|Z\|e^{j\theta}$
R		Crystal resistance, $R = \mathrm{Re}(Z)$
X		Crystal reactance, $X = \mathrm{Im}(Z)$
θ		Crystal impedance phase angle, $\theta = \arg(Z)$

[3] Strictly speaking, *RR* (or R_r) refers to the resistance at F_r, however as shown in Section 8, the distinction between *RR* and R_1 is rarely worthwhile.